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# A NEW THEOREM ON ELECTRICAL NETWORKS

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1. The theorem is based on a, apparently little known, remark of very general validity by Oliver Heaviside in his „Electrical Papers", Vol. II. page 412. In a typical Heaviside-way it was written down in the middle of an extensive theoretical investigation on electromagnetic waves. The late Professor H. A. Lorentz, who, as I often had the occasion to conclude from personal conversations, had a great admiration for the work of Heaviside, discovered this casual remark and he gave, late in life, a proof of it based on the Maxwellian equations [1]). This is the only reference I know of to the statement which Heaviside gave without any proof whatsoever. In the language of ordinary circuit theory it could be worded as follows:

Given a constant passive network at rest. If at the time $t = 0$ suddenly a constant E.M.F. is applied to the network, in general transients will occur. After a long time, say $t = t_1$, the transients may be considered to have died down and a direct current only will in general be present in the network. This direct current (if there is any) will cause a Joulean heat dissipation at a constant rate per second. If we call $W'$ „the pseudo heat dissipation" which would have occurred if the constant, final, current would have been present all the time from $t = 0$ to $t = t_1$, Heaviside's remark is to the effect, that, when the steady state has been reached, the total amount of work $A$ done by the E.M.F. exceeds the „pseudo heat dissipation" (if there is any) by twice the excess of the electric energy $U$ over the magnetic energy $T$, or:

$$A - W' = 2\,(U - T) \tag{1}$$

2. Another very simple new theorem, connecting the final steady

state with the a.c. admittance or impedance of the network can be derived from (1) with the aid of the modern form of the symbolic calculus [2]).

Let an E.M.F. of form [1] $V$ (zero before, $V$ after $t = 0$) be applied to a passive circuit originally at rest; let the current delivered by the E.M.F. be $i(t)$. The rules of the symbolic calculus give for $i(t)$:

$$i(t) \doteqdot p \int_0^\infty e^{-pt}\, i(t)dt = \frac{V}{Z(p)} = V \,.\, Y(p), \tag{2}$$

where $Z(j\omega)$ and $Y(j\omega)$ are the a.c. impedance and admittance resp., seen from the terminals of the E.M.F.

Now split up the current $i(t)$ in two parts:

$$i(t) = \bar{\imath} + i^*(t), \tag{3}$$

where $\bar{\imath}$ is the final steady value which obtains when all the transients have died down, and $i^*(t)$ can be considered as the transient part of $i(t)$, so that e.g. $i^*(\infty) = 0$.

Further the work $A$ done by the E.M.F. up to the time $t_1$ is given by

$$A = \int_0^{t_1} V \,.\, i(t)\, dt = V \,.\, \int_0^{t_1} i(t)dt.$$

Again, the above defined „pseudo heat dissipation" $W'$ can be written as

$$W' = V \int_0^{t_1} \bar{\imath}\, dt = t_1 V\, \bar{\imath}$$

Hence the difference $A - W'$ is given by

$$A - W' = V \int_0^{t_1} \{i(t) - \bar{\imath}\}\, dt = V \int_0^{t_1} i^*(t)dt,$$

and for $t_1 \to \infty$ this becomes

$$A - W' = V \int_0^\infty \{i(t) - \bar{\imath}\}\, dt. \tag{4}$$

Now, according to (2), the final steady current $\bar{\imath}$ is represented by

$$\bar{\imath} = V \,.\, Y(o), \tag{5}$$

so that instead of (4) we may write:

$$A - W' = V^2 \left[\int_0^\infty e^{-pt}\, \{i(t) - Y(o)\}\, dt\right]_{p=0}$$

or, according to (2):

$$A - W' = V^2 \left[\frac{1}{p}\, \{Y(p) - Y(o)\}\right]_{p=0} =$$

$$= V^2 \left[\frac{1}{p}\, \{Y(o) + pY'(o) + \frac{p^2}{2!}\, Y''(o) + \ldots\ldots - Y(o)\}\right]_{p=0} = V^2 .\, Y'(o).$$

Hence, with (1) we obtain our final result

$$\boxed{V^2 \cdot Y'(o) = 2(U - T),} \tag{6}$$

representing the

Theorem.

*If a constant E.M.F. of amount unity is suddenly applied to an arbitrary network originally at rest, the excess of the electric energy over the magnetic energy in the final steady state equals numerically half the differential of the admittance with respect to $j\omega$ at the frequency $\omega = 0$.*

*Remark 1.*

Elimination of $V^2$ from (5) and (6) yields

$$-\tfrac{1}{2} \cdot \frac{Y'(o)}{Y^2(o)} \cdot \bar{\imath}^2 = T - U.$$

Hence introducing the impedance $Z(p) = 1/Y(p)$, we obtain:

$$\tfrac{1}{2} Z'(o) \cdot \bar{\imath}^2 = T - U, \tag{7}$$

which represents another form of the theorem. (6) may, however, be preferred when $\bar{\imath} = 0$.

*Remark 2.*

Moreover, elimination from (6) of one factor $V$ yields, with the aid of (5)

$$\bar{\imath} V \cdot \frac{Y'(o)}{Y(o)} = 2\,(U - T). \tag{8}$$

But

$$\bar{\imath} V = \overline{W'},$$

$\overline{W'}$ being the heat dissipation per second in the final steady state; hence (8) can be written

$$-\frac{Y'(o)}{Y(o)} = \frac{Z'(o)}{Z(o)} = \frac{d}{dj\omega} \{\log Z(j\omega)\}_{j\omega=0} = \frac{2(T - U)}{\overline{W'}},$$

or, in words:

*The ratio of twice the excess of the magnetic energy over the electric energy to the heat dissipation per second, all in the final, steady state, equals the differential of the logarithm of the impedance with respect to $j\omega$ at the frequency $\omega = 0$.*

*Remark 3.*

When the system in the final state does not allow a direct current to be present ($\bar{\imath} = 0$), as e.g. is the case when a condenser is directly in series with the applied E.M.F., no magnetic energy will ultimately be present, because then all currents will be zero. Then

$$W' = T = 0$$

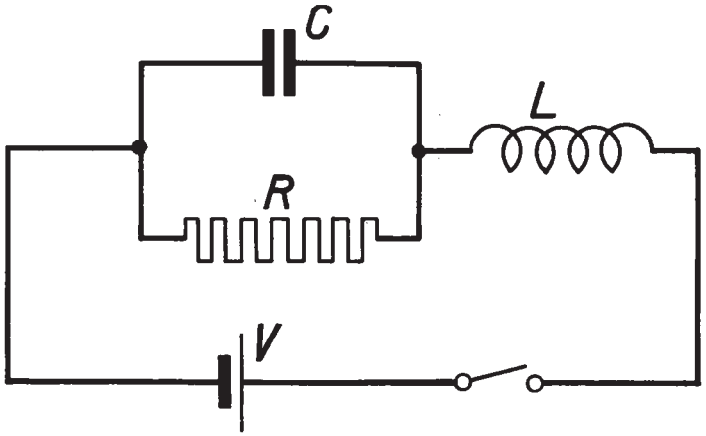
and therefore

$$U = \tfrac{1}{2}A,$$

which means that *the efficiency with which a battery can be made to charge a system of condensers (any resistances and inductances being present in the circuit) is always 50%.*

*Example.*

Consider the circuit of the figure.




The impedance obviously is

$$Z(j\omega) = j\omega L + \frac{R \cdot \frac{1}{j\omega C}}{R + \frac{1}{j\omega C}},$$

so that the admittance becomes:

$$Y(j\omega) = \frac{1}{Z(j\omega)} = \frac{1 + j\omega CR}{R + j\omega L + (j\omega)^2 LCR}.$$

Differentiating with respect to $j\omega$, and putting $\omega = 0$, yields

$$\left(\frac{dY(j\omega)}{d(j\omega)}\right)_{\omega=0} = C - \frac{L}{R^2}.$$

Hence:

$$V^2 . Y'(o) = CV^2 - \frac{L}{R^2} V^2 = CV^2 - L\bar{i}^2,$$

which equals twice the difference between the electric energy and the magnetic energy in the final steady state, as required by the theorem (6).

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REFERENCE

1) H. A. Lorentz, Collected Works, Vol. III, page 331 or: Proc. Nat. Acad. of Sciences, **8**, 333, 1922.
2) Phil. Mag., **7**, 1153, 1929.